F. Corks

Freedman showed that two closed, simply connected, smooth 4-manifolds are

homotopy equivalent
$\Leftrightarrow$
homeomorphic

(if not smooth the this is not quite right)

but Donaldson + Freedman $\Rightarrow \exists$ infinitely many distinct smooth manifolds that are all homeomorphic!

Big Question: how do you organize this mess of smooth manifolds?

So far no one knows how to do this, but here is a curious result that might help

Th^{m} 14:

let X_0, X_1 be closed, simply connected, smooth 4-manifolds that are homeomorphic

Then $\exists$ a smooth contractible manifold

$$C \subset X_0$$

and an involution

$$\phi : \partial C \to \partial C$$

such that

$$X_1 = \overline{(X_0 - C)} \cup_\phi C$$

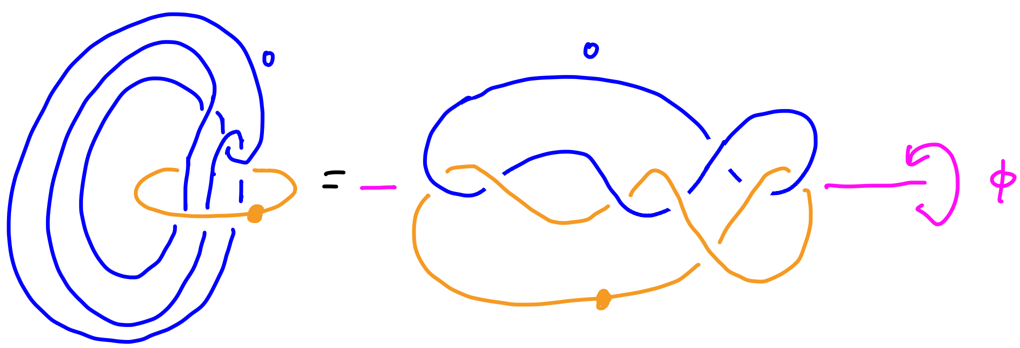
C is called a cork

so th^{m} says homeomorphic non-diffeomorphic, simply connected manifolds differ by a <u>cork twist</u>

Remark:

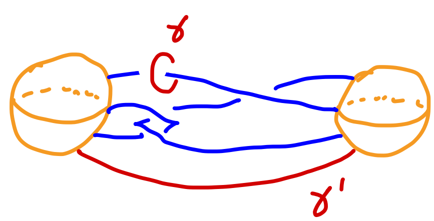
1) clearly diffeomorphism $\phi: \partial C \to \partial C$ can't be extended over C (or else X_1, X_2 diffeo!)

2) Akbulut found the first cork
it was the Mazu manifold!

$C =$ 

involution ϕ is rotation about axis

detour if you know contact/symplectic geometry



- the orange and blue define a Stein 4-manifold
- ϕ takes γ to γ'
- γ bounds a disk in C (the cocore of handle)
- $tb(\gamma') = 0$
- slice-Bennequin inequality implies

$tb(\gamma') \leq -\chi(\text{surface in } C \text{ w/ } \partial = \gamma')$

so γ' does not bound a disk in C

- therefore ϕ does not extend over C

3) There has been lots of recent work

Tange: $\exists$ C and $\phi: \partial C \to \partial C$ of order n

s.t. $C \hookrightarrow X \leftarrow$ some 4-mfd

and $\overline{(X-C)} \cup_{\phi^k} C$ all

different for $0 \leq k < n$

Auckley-Kim-Melvin-Ruberman:

if G is any finite subgroup of $SO(4)$

then $\exists$ C and a G-action on ∂C

s.t. $\overline{(X-C)} \cup_g C$ all distinct

Gompf-Akbulut: "infinite order corks"

Thm 15:

let W be a 5-dimensional h-cobordism between simply connected closed 4-manifolds X_0 and X_1

Then $\exists$ a subcobordism $A \subset W$ from $A_0 \subset X_0$ to $A_1 \subset X_1$ such that

1' A_0, A_1, A are compact contractible manifolds

2' $W - \text{int}\, A$ is a product cobordism

i.e. $\cong [0,1] \times (X_0 - \text{int}\, A_0)$
$\cong [0,1] \times (X_1 - \text{int}\, A_1)$

3' $W - A$ is simply connected

④[3] $A \cong B^5$

⑤[2] $[0,1]\times A_0 \cong [0,1]\times A_1 \cong B^5$

*(see below)

⑥[2] $A_0 \cong A_1$ diffeo. restricted to ∂ is an involution

1 Curtis-Freedman-Hsiang-Stong '96
2 Matveyev '96
3 Bižaca unpublished

Proof of Th^{m} 14:

given X_0, X_1 as in the theorem, we know there is an h-cobordism W by Th^{m} 12

now by ② $\overline{X_0 - A_0} \cong \overline{X_1 - A_1}$

by ⑥ $A_0 \cong A_1$ so X_0 and X_1 agree in the complement of $C := A_0 \cong A_1$

note*: $\partial A_0 = \partial A_1$ by ② so diffeo in ⑥ restricted to ∂A_0 can be thought of as a diffeo of ∂A_0 to itself

we will see that this is the involution taking X_0 to X_1 ∎

Proof of 15: given W we can assume it has a handle decomposition with no 0 or 5 handles

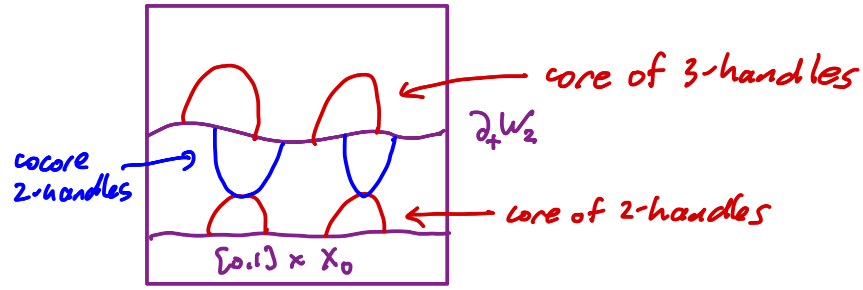
and as in the proof of the h-cobordism th^{m} (Th^{m} II.1)

we can exchange 1 for 3-handles and 4 for 2-handles

so

$W = ([0,1]\times X_0) \cup$ 2-handles $\cup$ 3-handles

recall $W_k = \cup$ of all handles $\leq k$ (and $[0,1]\times X_0$)



in $\partial_+ W_2$ let

$S_{0,i}$ $i = 1, \dots, k$ be the belt spheres of 2-handles

$S_{1,i}$ $i = 1, \dots, k$ be the attaching spheres of 3-handles

since W is an h-cobordism we can number spheres and possibly handle slide so that $S_{0,i} \bullet S_{1,j} = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

algebraic intersection

we can assume all $S_{i,j}$ are transverse to eachother

choose a base point $*$ in $\partial_+ W_2$ disjoint from spheres

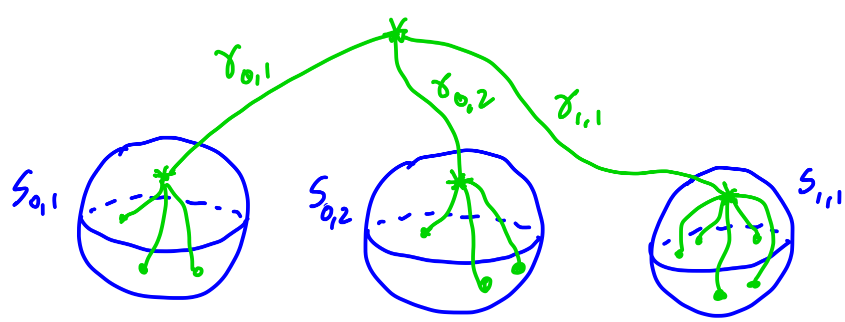
and base point $*_{i,j}$ on $S_{i,j}$

let $\gamma_{i,j}$ be disjoint arcs in $\partial_+ W_2$ from $*$ to $*_{i,j}$

(only intersecting $S_{i,j}$ in end points)

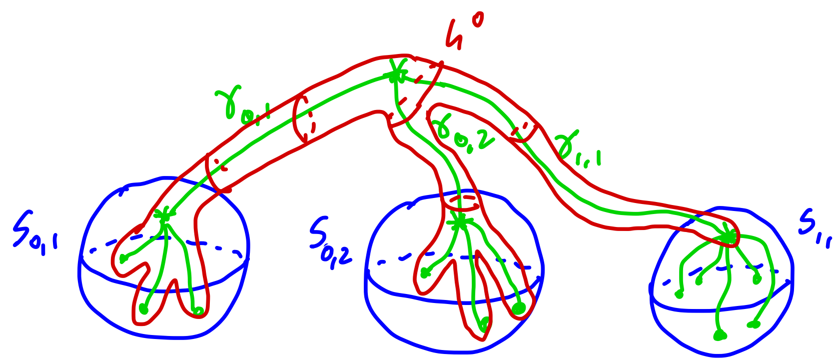
let η_α be with disjoint interiors arcs in $S_{i,j}$

connecting $*_{ij}$ to intersection points of $S_{i,j}$ with other $S_{k,l}$



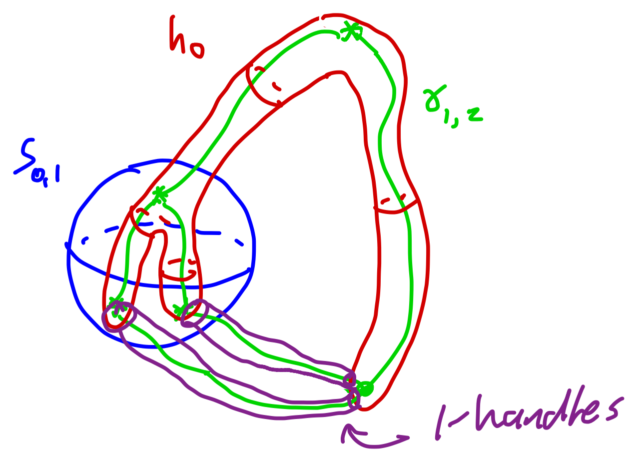
note: small nbhd of $\left(\bigcup \gamma_{i,j} \cup \bigcup \eta_\alpha \text{ in } S_{0,i}\text{'s}\right)$

$\cong B^4$ think of as a 0-handle



now nbhds of η_α in $S_{1,i}$'s are attached to h^0

as 1-handles



note a nbhd of $S_{i,j}$ outside 0-h $\cup$ 1-handles

is attached as a 2-handle

so we have $\underbrace{h^2_1, \ldots, h^2_k}_{\text{from } S_{0,i}}, \underbrace{h^2_{k+1} \ldots h^2_{2k}}_{\text{from } S_{1,i}}$

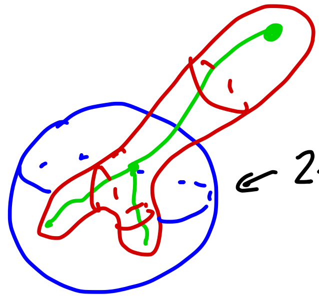
and a nbhd N of (arcs and spheres)

is 0-h $\cup$ 1-h's $\cup$ 2-h's

note: 1) $h^2_1 \ldots h^2_k$ don't go over 1-handles

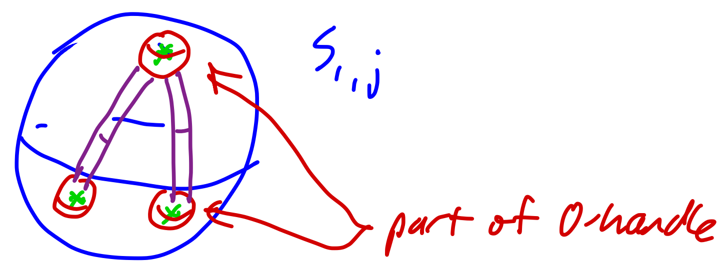
and are attached to unknots

e.g.



$\leftarrow$ 2-handle only attached to h^0

2) h^2_j for $j > k$ do go over 1-handles but in a very special way



so $\pi_1(N) = \langle x_1, \dots, x_\ell \mid r_1, \dots, r_{2k} \rangle$

one for each 1-handle

1 for each 2-handle

<u>note</u>: $r_1, \dots, r_k$ trivial since the h^2_i $i \leq k$ don't go over 1-handles

$r_{k+1}, \dots r_{2k}$ are products of $x_i \bar{x}_i$ or $\bar{x}_i x_i$.

so they are trivial too!

now extend handle str above on N to all of $\partial_+ W_2$

so we add 1-handles $\tilde{h}^1_1 \dots \tilde{h}^1_{k'}$

2-handles $\tilde{h}^2 \dots \tilde{h}^2_{\ell'}$

3-handles and one 4-handle

$\pi_1(\partial_+ W_2) = \{1\}$

but has generators $h_1', \dots, h_\ell', \tilde{h}_1', \dots \tilde{h}_k'$

and relations coming from h_i^2 and $\tilde{h}_i^2$

these are trivial

there are "Tietze moves" that take the presentation of $\pi_1(\partial_+ W_2)$ to the trivial presentation

generators

given a presentation $\langle X \mid R \rangle$ relations

the moves are

1) add a relation:

normal closure of R

if $r \in N(R)$

then new presentation is

$\langle X \mid R \cup \{r\} \rangle$

2) remove a relation:

if $R' = R - \{r\}$ and $r \in N(R')$

then new presentation is

$\langle X \mid R' \rangle$

3) add a generator:

if g is a new symbol not in X

and w is any word in X

then new presentation is

$\langle X \cup \{g\} \mid R \cup \{wg^{-1}\} \rangle$

4) remove superflous generator:

if $g \in X$ and $wg^{-1} \in R$ for some word w in $X - \{g\}$

then the new presention is

$\langle X - \{g\} \mid R - \{wg^{-1}\} \rangle$

any 2 presentations of the same group are related by 1) - 4)

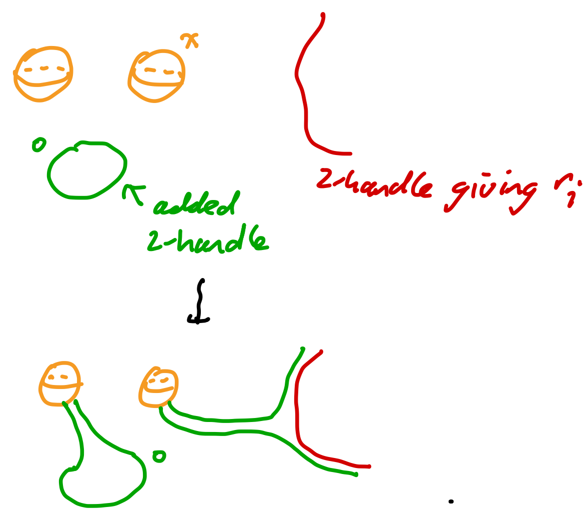
let's see how 1) - 4) related to handle bodies ($\geq 4D$)

1) given $r \in N(R)$ then r = product of conjugates in R

say $w_1 r_1 w_1^{-1} \cdots w_m r_m w_m^{-1}$

now add 2/3 cancelling pair
push attaching sphere over 1-handles to see $w_1 w_1^{-1}$ and then over 2-handle giving r_1, now continue other $w_1 r_1 w_1^{-1}$

eg



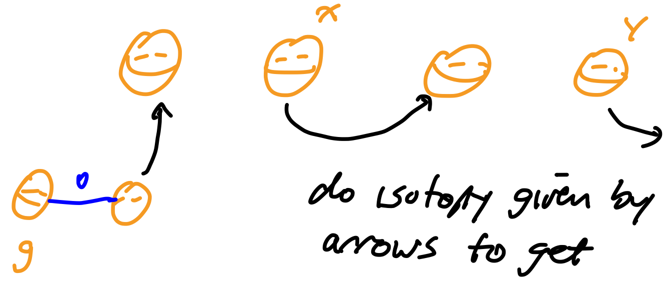
now green gives $x r_1 x^{-1}$

2) just ignor a 2-handle giving an extra rel^{n}

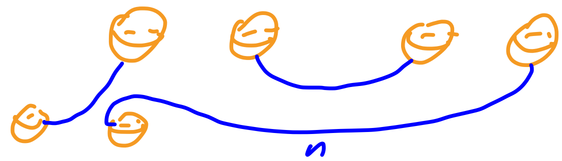
3) given w a word in X ← corresp to 1-handles

add cancelling 1/2 pair call new generator for 1-handle g now push 1-handle over other 1-handles corresponding to w

for example



do isotopy given by arrows to get

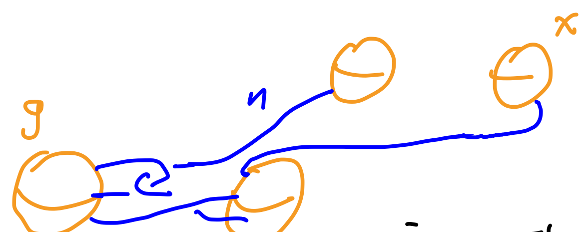


2-handle now gives xyg^{-1}

4) if $g \in X$ and $wg^{-1} \in R$

then 2-handle corresp. to wg^{-1} algebraically cancels (but not necessarily geometrically)

eg



gives xg^{-1}

so leave the handles but can remove g and xg^{-1} from presentation

thus we can do handle slides, add 1/2 and 2/3 cancelling pairs as dictated by the Tietze moves so that the handles $\tilde{h}^2_1, \ldots \tilde{h}^2_n$ cancel the generators h^1_i and $\tilde{h}^1_j$ and the handles $\tilde{h}^1_{n+1}, \ldots \tilde{h}^1_\ell$ give trivial rel$^{\underline{n}}$s let B_1 = all the 0 and 1-handles of $\partial_+ W_2 \cup (\tilde{h}_1^2 \dots \tilde{h}_n^2)$

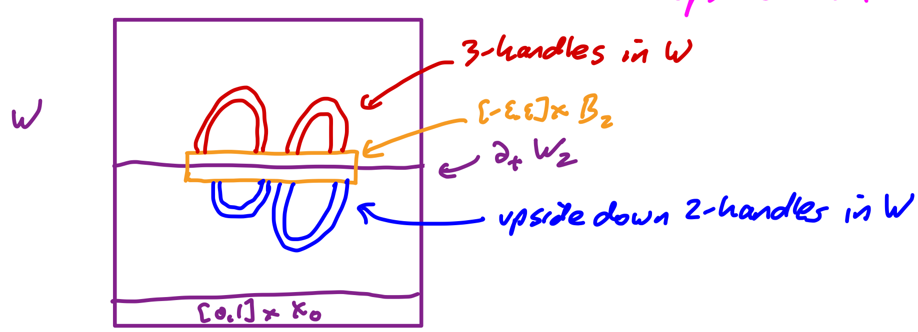
by construction $\pi_1(B_1) = 1$ and $\tilde{H}_2(B_1) = 0$

so B_1 is contractible

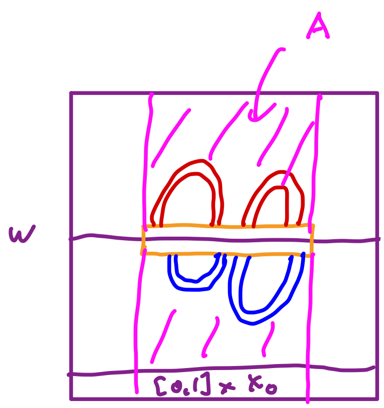
let $B_2 = B_1 \cup h_1^2 \cup \dots \cup h_{2k}^2$ (so $N \cup$ some $\tilde{h}_i^2$'s)

$B_3 = ([-\varepsilon,\varepsilon] \times B_2) \cup$ 3-handles attached to $S_{0,i}$ and $S_{1,i}$

upside down 2-handles



let $A = B_3 \cup$ (all of W above and below B_3)



② of Th$^{\underline{m}}$

<u>note</u>: 1) $M - A$ has no handles so is a product

2) $B_1 \times [0,1]$ is diffeomorphic to A since the added 2 and 3-handles cancel

so A is contractible

now $B_1 \times [0,1]$ is a 5-manifold with the "same" handle structure as B_1

in B_1 the attaching S^1's of the 2-handles are homotopic to generators of π_1 in ∂(1-handle body)

so in $B_1 \times [0,1]$ the are attached to a 4-manifold and so are isotopic to generators

$\therefore$ the 1 and 2-handles cancel

and we see $B_1 \times [0,1] \cong A$ is B^4 ④ in Th$^{\underline{m}}$

to see $A_0 \times [0,1] \cong B^4$ ⑤ in Th$^{\underline{m}}$ we need a handle decomposition of A_0

to get from B_2 to A_0 we attach 3-handles to $S_{0,i}$'s

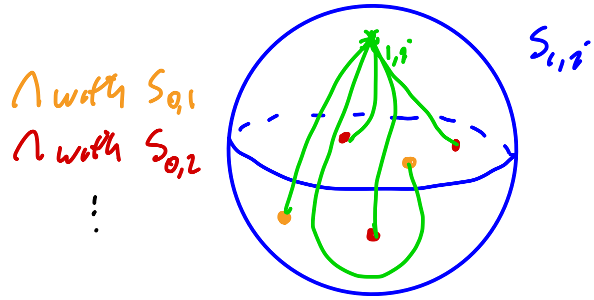
that means A_0 is obtained from B_2 by surgering the $S_{0,i}$'s from 2-handles to 1-handles

the new 1-handles give new generators to $\pi_1(A_0)$

call them $y_1, \ldots, y_n$

but the 2-handles $h^2_{k+1} \cdots h^2_{2k}$ (from $S_{1,i}$'s) give relations that kill the $y_1 \cdots y_n$

now when choosing arcs in $S_{1,i}$ we first connect $*_{1,i}$ to all intersections with $S_{0,1}$ then $S_{0,2}$, and so on



we can now see the rel^{n} from the 2-handle coming from $S_{1,i}$ is

$$w_1 w_2 \cdots w_k$$

where w_j is a word in y_j and $\bar{y}_j$

$$\text{with exponent sum} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$$

now as we saw for the Mazur manifold in $A_0 \times [0,1]$ where $h^2_{k+1} \cdots h^2_{2k}$ are attached to circles in a 4-manifold (where for S^1's, homotopy $\Rightarrow$ isotopy) the h^2_i's cancel the 1-handles

and $A_0 \times [0,1] \cong B_1 \times [0,1] \cong B^5$

can argue similarly for $A_1 \times [0,1]$

note: (5) also $\Rightarrow$ A_0 and A_1 contractible

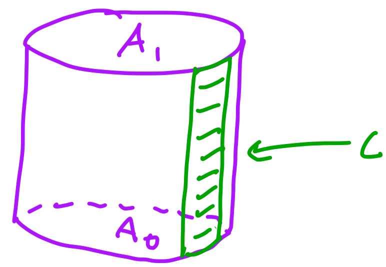
that and (4) $\Rightarrow$ (1) of Th$^{\underline{m}}$

now for (6)

note: $A \cong B^5$ so $S^4 = \partial A = A_0 \cup_\partial A_1$

thus $A_0 \cup_\partial A_1 - C \cong B^4$

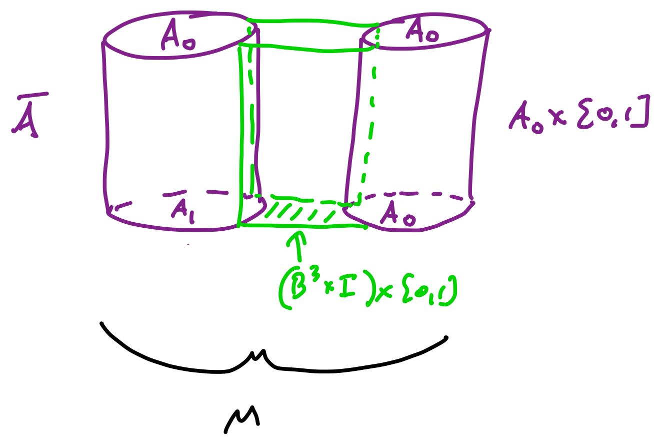
$C = [0,1] \times B^3 \subset \partial A$ with $B^3 \subset \partial A_0$



and similarly we know $A_0 \times [0,1] \cong B^5$ so

$$(A_0 \cup_\partial A_0) - C \cong B^4$$

consider $M = \bar{A} \cup (A_0 \times [0,1])$ glued along $B^3 \times [0,1]$



we know $\bar{A} \cong B^5$ and $A_0 \times [0,1] \cong B^5$

so $M \cong B^5$

and $\partial_- M = (A_0 \cup_\partial A_0) - C \cong B^4$

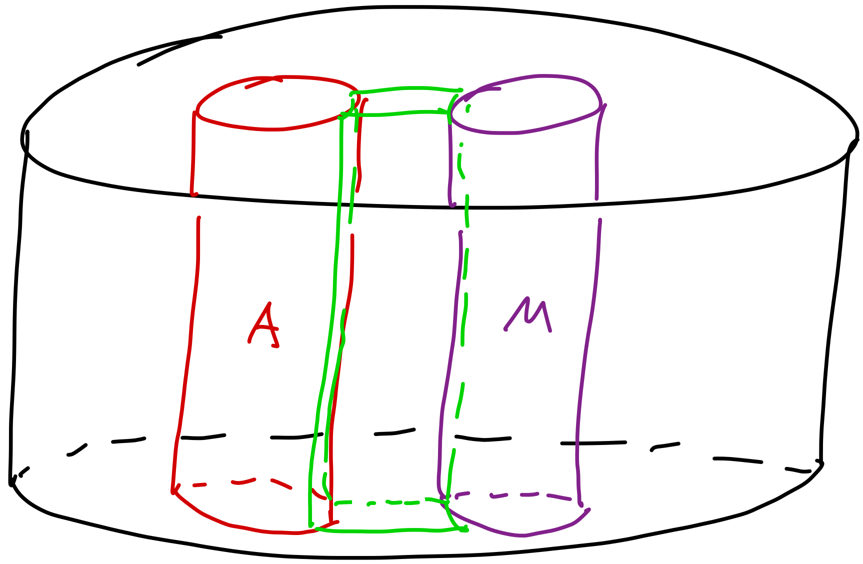
$\partial_+ M = (A_1 \cup_\partial A_0) - C \cong B^4$

so M is a cobordism $\cong ([0,1] \times B^4, B_0^4, B_1^4)$

$A_1 \cup_{B^3} A_0$ $\quad$ $A_0 \cup_{B^3} A_0$

we can now insert this in the product part of W

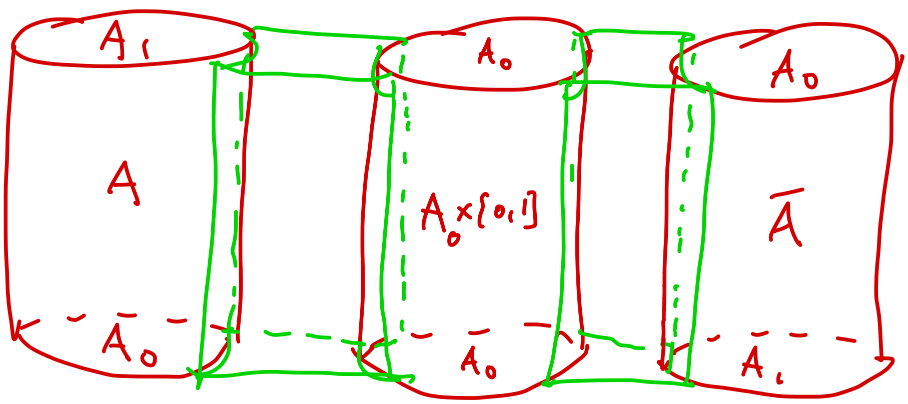
so we see



rename A to all of this

all previous properties hold but now

we see



now there is an obvious involution taking

top to bottom and X_1 differs from X_0

by this involution

we only have 3 left. This is a little involved and not used much(?), so we refer to

Kirby "Akbulut's corks and h-cobordisms of smooth, simply connected 4-mfds"